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SOLUTIONS OF THE EQUATIONS OF RADIATIVE TRANSFER BY AN INVARIANT IMBEDDING APPROACH

C. N. Adams and G. W. Kattawar

Department of Physics, Texas A & M University, College Station, Texas 77843

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Abstract—The nonlinear integro-differential equations satisfied by the scattering and transmission matrices for scattering according to a Rayleigh phase matrix are accurately solved numerically. The method is easily programmed and avoids problems of unicity which are encountered when solutions are obtained using Chandrasekhar's *X*- and *Y*-functions. The reflected and transmitted intensities for scattering according to both a Rayleigh phase function and a Rayleigh phase matrix are compared for a wide range of incident solar angles, ground albedos, and optical depths up to $\tau = 5$. The solutions have been compared whenever possible with other solutions obtained by using the *X*- and *Y*-functions and the agreement is excellent.

INTRODUCTION

The power in the invariant imbedding approach is that it replaces a system of linear integro-differential equations having two point boundary conditions with a system of non-linear integro-differential equations possessing a set of initial conditions. This latter set of equations is ideally suited for numerical calculations. With the conventional approach employing the *X*- and *Y*-functions, problems of unicity of the solutions are prevalent as was shown by Chandrasekhar[1] and Mullikin.[2] Bellman *et al.*[3,4] successfully used the invariant imbedding technique for the solution of problems in radiative transfer in slabs of finite thickness with isotropic scattering. Kagiwada and Kalaba[5] have numerically solved the scalar equation of transfer for an atmosphere with a single scattering phase function of the form $p(\mu) = 1 + \mu$. This solution was accomplished by a Fourier expansion of the single scattering phase function in terms of associated Legendre polynomials. Similar expansions for the scattering and transmission yield Fourier coefficients with no explicit azimuth dependence. The resulting integro-differential equations have initial conditions which render them easily solvable by using highly accurate numerical integration schemes in double precision. It should also be noted that when one obtains a solution corresponding to a certain depth, the solutions at all intermediate optical depths consistent with the step size are also obtained.

The purpose of this paper is twofold. First we wish to demonstrate the practicality of the invariant-embedding technique as applied to the much more difficult problem for scattering according to a Rayleigh phase matrix. Secondly, we will compare the results between the correct phase matrix formulation and the phase function formulation using the same numerical techniques. This will allow some very good comparisons between the two and we will show that very large errors can occur when using the Rayleigh phase function as opposed to the correct formulation employing the Rayleigh phase matrix.

SCATTERING ACCORDING TO A RAYLEIGH PHASE FUNCTION

Using the notation of CHANDRASEKHAR,[1] the equation for the scattering and transmission functions for a general law of scattering with single scattering albedo λ are respectively:

$$\frac{\partial S(\tau_1;\mu,\phi:\mu_0,\phi_0)}{\partial\tau_1}+\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S(\tau_1;\mu,\phi;\mu_0,\phi_0)=\lambda(\tau_1)\Bigg\{p(\mu,\phi;-\mu_0,\phi_0)$$
$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}p(\mu,\phi;\mu'',\phi'')S(\tau_1;\mu'',\phi'',\mu_0,\phi_0)\frac{d\mu''}{\mu''}\,d\phi''$$
$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}S(\tau_1;\mu,\phi;\mu',\phi')p(-\mu',\phi';-\mu_0,\phi_0)\frac{d\mu'}{\mu'}\,d\phi'$$
$$+\frac{1}{(4\pi)^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}S(\tau_1;\mu,\phi;\mu',\phi')p(-\mu',\phi':\mu'',\phi'')$$
$$\times S(\tau_1;\mu'',\phi'';\mu_0,\phi_0)\frac{d\mu'}{\mu'}\,d\phi'\frac{d\mu''}{\mu''}\,d\phi''\Bigg\}. \tag{1}$$

$$\frac{1}{\mu_0}T(\tau_1;\mu,\phi;\mu_0,\phi_0)+\frac{\partial T(\tau_1;\mu,\phi;\mu_0,\phi_0)}{\partial\tau_1}=\lambda(\tau_1)\Bigg\{e^{-\tau_1/\mu}p(-\mu,\phi;-\mu_0,\phi_0)$$
$$+\frac{e^{-\tau_1/\mu}}{4\pi}\int_0^1\int_0^{2\pi}p(-\mu,\phi;\mu'',\phi'')S(\tau_1;\mu'',\phi'';\mu_0,\phi_0)\frac{d\mu''}{\mu''}\,d\phi''$$
$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}T(\tau_1;\mu,\phi;\mu',\phi')p(-\mu',\phi':-\mu_0,\phi_0)\frac{d\mu'}{\mu'}\,d\phi'$$
$$+\frac{1}{(4\pi)^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}T(\tau_1;\mu,\phi;\mu',\phi')p(-\mu',\phi';\mu'',\phi'')$$
$$\times S(\tau_1;\mu'',\phi'';\mu_0,\phi_0)\frac{d\mu'}{\mu'}\,d\phi'\frac{d\mu''}{\mu''}\,d\phi''\Bigg\}. \tag{2}$$

If we assume that the phase function can be expanded in a finite number of Legendre polynomials, say N:

$$p(\cos\theta)=\sum_{l=0}^{N}\omega_l P_l(\cos\theta) \tag{3}$$

and by using the addition theorem for spherical harmonics we get

$$p(\mu,\phi;\mu',\phi')=\sum_{m=0}^{N}(2-\delta_{om})\left\{\sum_{l=m}^{N}\omega_l^m P_l^m(\mu)P_l^m(\mu')\right\}\cos m(\phi'-\phi) \tag{4}$$

where

$$\omega_l^m = \omega_l \frac{(l-m)!}{(l+m)!} \qquad (l = m, \ldots, N, 0 \le m \le N) \tag{5}$$

and δ_{om} is the Kronecker delta.

When the single scattering function has the form of equation (4) then the scattering and transmission functions have similar expansions, namely

$$\begin{aligned} S(\tau_1;\mu,\phi;\mu_0,\phi_0) &= \sum_{m=0}^{N} S^{(m)}(\tau_1;\mu,\mu_0)\cos m(\phi_0-\phi) \\ T(\tau_1;\mu,\phi;\mu_0,\phi_0) &= \sum_{m=0}^{N} T^{(m)}(\tau_1;\mu,\mu_0)\cos m(\phi_0-\phi). \end{aligned} \tag{6}$$

Substituting equation (6) into equations (1) and (2) we get

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S^{(m)}(\tau_1;\mu,\mu_0)+\frac{\partial S^{(m)}(\tau_1;\mu,\mu_0)}{\partial \tau_1} = \lambda(2-\delta_{om})\sum_{l=m}^{N}(-1)^{m+l}\omega_l^m\psi_l^m(\tau_1;\mu)\psi_l^m(\tau,\mu_0) \tag{7}$$

$$\frac{1}{\mu_0}T^{(m)}(\tau_1;\mu,\mu_0)+\frac{\partial T^{(m)}(\tau_1;\mu,\mu_0)}{\partial \tau_1} = \lambda(2-\delta_{om})\sum_{l=m}^{N}\omega_l^m\phi_l^m(\tau_1;\mu)\psi_l^m(\tau_1;\mu_0) \tag{8}$$

where

$$\psi_l^m(\tau_1;\mu) = P_l^m(\mu)+\frac{(-1)^{l+m}}{2(2-\delta_{om})}\int_0^1 S^{(m)}(\tau_1;\mu,\mu')P_l^m(\mu')\frac{d\mu'}{\mu'} \tag{9}$$

and

$$\phi_l^m(\tau_1;\mu) = e^{-\tau_1/\mu}P_l^m(\mu)+\frac{1}{2(2-\delta_{om})}\int_0^1 T^{(m)}(\tau_1;\mu,\mu')P_l^m(\mu')\frac{d\mu'}{\mu'}. \tag{10}$$

Let us now note that for Rayleigh scattering $p(\cos\theta) = \frac{3}{4}(1+\cos^2\theta)$ and we easily get that $\omega_0 = 1, \omega_1 = 0, \omega_2 = \frac{1}{2}$, and $N = 2$.

Equations (7) and (8) have the following initial conditions

$$S^{(m)}(0,\mu,\mu_0) = 0, \qquad m = 0,1,\ldots,N \tag{11}$$

$$T^{(m)}(0,\mu,\mu_0) = 0, \qquad m = 0,1,\ldots,N \tag{12}$$

We thus have a set of $2(N+1)$ nonlinear integro-differential equations to solve with simple initial conditions. To put the equations in a form amenable to numerical techniques the integrals were represented by a Gaussian quadrature of order M with weights w_i and shifted roots $\mu_i (i = 1, 2, \ldots, m)$. Therefore the discrete form for the scattering and transmission function can be written as

$$S_{ij}^{(m)}(\tau_1) = S^{(m)}(\tau_1;\mu_i,\mu_j) \tag{13}$$

$$T_{ij}^{(m)}(\tau_1) = T^{(m)}(\tau_1;\mu_i,\mu_j). \tag{14}$$

Using this notation the discrete forms of equations (7–10) become

$$\left(\frac{1}{\mu_i}+\frac{1}{\mu_j}\right)S_{ij}^{(m)}+\frac{dS_{ij}^{(m)}}{d\tau_1} = \lambda(2-\delta_{om})\sum_{l=m}^{N}(-1)^{l+m}\omega_l^m\psi_{li}^m\psi_{lj}^m \tag{15}$$

$$\frac{1}{\mu_j}T_{ij}^{(m)}+\frac{dT_{ij}^{(m)}}{d\tau_1} = \lambda(2-\delta_{om})\sum_{l=m}^{N}\omega_l^m\phi_{li}^m\psi_{lj}^m \tag{16}$$

where

$$\psi_{li}^m = P_l^m(\mu_i)+\frac{(-1)^{l+m}}{2(2-\delta_{om})}\sum_{k=1}^{M}S_{ik}^{(m)}P_l^m(\mu_k)\frac{w_k}{\mu_k} \tag{17}$$

and

$$\phi_{li}^m = e^{-\tau_1/\mu_i}P_l^m(\mu_i)+\frac{1}{2(2-\delta_{om})}\sum_{k=1}^{M}T_{ik}^{(m)}P^m(\mu_k)\frac{w_k}{\mu_k}. \tag{18}$$

The initial conditions are simply

$$S_{ij}^{(m)}(0) = 0, \qquad T_{ij}^{(m)}(0) = 0. \tag{19}$$

From equations (15) and (16) we see that there are $2M^2(N+1)$ equations to be solved: however by using the principle of reciprocity, i.e. $S_{ij}^{(m)} = S_{ji}^{(m)}$ and $T_{ij}^{(m)} = T_{ji}^{(m)}$, this number can be reduced to $M(M+1)(N+1)$.

Once S and T have been calculated the reflected and transmitted intensities are easily obtained as follows.

$$I(0,\mu,\phi) = \frac{F_0}{4\mu}S(\tau_1;\mu,\phi;\mu_0,\phi_0) \tag{20}$$

and

$$I(\tau_1,-\mu,\phi) = \frac{F_0}{4\mu}T(\tau_1;\mu,\phi;\mu_0,\phi_0). \tag{21}$$

The numerical solutions to equations (15) and (16) will be discussed in the section entitled computational methods.

SCATTERING ACCORDING TO A RAYLEIGH PHASE MATRIX

It was pointed out by CHANDRASEKHAR[1] that if one is to treat the radiation field properly then it should of course be regarded as a vector field rather than a scalar field. This formulation leads to considering the transformation properties of the Stokes vector. For the case of scattering according to a Rayleigh phase matrix, SEKERA[6] has formulated the equations for the scattering and transmission matrices. Employing the notation of SEKERA[6] with only obvious modifications we proceed as follows. Let $\mathbf{S}(\tau_1;\mu,\phi;\mu_0,\phi_0)$

and $\mathbf{T}(\tau_1;\mu,\phi;\mu_0,\phi_0)$ be the scattering and transmission matrices then Sekera shows.

$$\left\{\frac{1}{\mu_0}+\frac{1}{\mu}+\frac{\partial}{\partial\tau_1}\right\}\mathbf{S}(\tau_1;\mu,\phi;\mu_0,\phi_0) = \lambda(\tau_1)\{\mathbf{P}(\mu,\phi;-\mu_0,\phi_0)\}$$
$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}\mathbf{P}(\mu,\phi;\mu'',\phi'')\mathbf{S}(\tau_1;\mu'',\phi'';\mu_0,\phi_0)\frac{d\Omega''}{\mu''}$$
$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}\mathbf{S}(\tau_1;\mu,\phi;\mu',\phi')\mathbf{P}(-\mu',\phi';-\mu_0,\phi_0)\frac{d\Omega'}{\mu'}$$
$$+\left(\frac{1}{4\pi}\right)^2\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}\mathbf{S}(\tau_1;\mu,\phi;\mu',\phi')\mathbf{P}(-\mu',\phi';\mu'',\phi'')\mathbf{S}(\tau_1;\mu'',\phi'';\mu_0,\phi_0)\frac{d\Omega''}{\mu''}\frac{d\Omega'}{\mu'} \quad (22)$$

and

$$\left\{\frac{1}{\mu_0}+\frac{\partial}{\partial\tau_1}\right\}\mathbf{T}(\tau_1;\mu,\phi;\mu_0,\phi_0) = \lambda(\tau_1)\{\mathbf{P}(-\mu,\phi;\mu_0,\phi_0)\,e^{-\tau_1/\mu}$$
$$+\frac{1}{4\pi}e^{-\tau_1/\mu}\int_0^1\int_0^{2\pi}\mathbf{P}(-\mu,\phi;\mu'',\phi'')\mathbf{S}(\tau_1;\mu'',\phi'';\mu_0,\phi_0)\frac{d\Omega''}{\mu''}$$
$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}\mathbf{T}(\tau_1;\mu,\phi;\mu',\phi')\mathbf{P}(-\mu',\phi';-\mu_0,\phi_0)\frac{d\Omega'}{\mu'} \quad (23)$$
$$+\left(\frac{1}{4\pi}\right)^2\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}\mathbf{T}(\tau_1;\mu,\phi;\mu',\phi')\mathbf{P}(-\mu',\phi';\mu'',\phi'')\mathbf{S}(\tau_1;\mu'',\phi'';\mu_0,\phi_0)\frac{d\Omega''}{\mu''}\frac{d\Omega'}{\mu'}$$

where $d\Omega = d\mu\, d\phi$. For Rayleigh scattering the phase matrix can be written in the following form.

$$\mathbf{P}(\mu,\phi;\mu_0,\phi_0) = \mathbf{P}^{(0)}(\mu,\mu_0)+(1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}\mathbf{P}^{(1)}(\mu,\phi;\mu_0,\phi_0)+\mathbf{P}^{(2)}(\mu,\phi;\mu_0,\phi_0) \quad (24)$$

$$\mathbf{P}^{(0)}(\mu,\mu_0) = \tfrac{3}{4}\begin{pmatrix} 2(1-\mu^2)(1-\mu^2)+\mu^2\mu_0^2 & \mu^2 & 0 \\ \mu_0^2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (25)$$

$$\mathbf{P}^{(1)}(\mu,\phi;\mu_0,\phi_0) = \tfrac{3}{4}\begin{pmatrix} 4\mu\mu_0\cos(\phi_0-\phi) & 0 & 2\mu\sin(\phi_0-\phi) \\ 0 & 0 & 0 \\ -4\mu_0\sin(\phi_0-\phi) & 0 & 2\cos(\phi_0-\phi) \end{pmatrix} \quad (26)$$

$$\mathbf{P}^{(2)}(\mu,\phi;\mu_0,\phi_0) = \tfrac{3}{4}\begin{pmatrix} \mu^2\mu_0^2\cos 2(\phi_0-\phi) & -\mu^2\cos 2(\phi_0-\phi) & \mu^2\mu_0\sin 2(\phi_0-\phi) \\ -\mu_0^2\cos 2(\phi_0-\phi) & \cos 2(\phi_0-\phi) & -\mu_0\sin 2(\phi_0-\phi) \\ -2\mu\mu_0^2\sin 2(\phi_0-\phi) & 2\mu\sin 2(\phi_0-\phi) & 2\mu\mu_0\cos 2(\phi_0-\phi) \end{pmatrix}. \quad (27)$$

In an analogous way **I**, **S**, and **T** can be written

$$\mathbf{I}(\tau;\mu,\phi) = \mathbf{I}^{(0)}(\tau;\mu,\mu_0)+(1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}\mathbf{I}^{(1)}(\tau;\mu,\mu_0;\phi_0-\phi) + \mathbf{I}^{(2)}(\tau;\mu,\mu_0;2(\phi_0-\phi)) \tag{28}$$

$$\mathbf{N}(\tau;\mu,\phi;\mu',\phi') = \mathbf{N}^{(0)}(\tau;\mu,\mu')+(1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\mathbf{N}^{(1)}(\tau;\mu,\mu';\phi'-\phi) + \mathbf{N}^{(2)}(\tau;\mu,\mu';2(\phi'-\phi)) \tag{29}$$

where ($\mathbf{N} = \mathbf{S}, \mathbf{T}$).

SEKERA[6] shows that the vector equations involving **S** and T can be reduced to the following scalar equations,

$$\left\{\frac{1}{\mu_0}+\frac{1}{\mu}+\frac{\partial}{\partial\tau_1}\right\}S_k(\tau_1;\mu,\mu_0) = \lambda(\tau_1)\left\{1+\int_0^1 \psi^k(\mu')S_k(\tau_1;\mu,\mu')\frac{d\mu'}{\mu'}\right\} \times\left\{1+\int_0^1 \psi^k(\mu'')S_k(\tau_1;\mu_0,\mu'')\frac{d\mu''}{\mu''}\right\} \quad (k=1,2) \tag{30}$$

$$\left\{\frac{1}{\mu_0}+\frac{\partial}{\partial\tau_1}\right\}T_k(\tau_1;\mu,\mu_0) = \lambda(\tau_1)\left\{e^{-\tau_1/\mu}+\int_0^1 \psi^k(\mu')T_k(\tau_1;\mu,\mu')\frac{d\mu'}{\mu'}\right\} \times\left\{1+\int_0^1 \psi^k(\mu')S_k(\tau_1;\mu,\mu')\frac{d\mu'}{\mu'}\right\} \quad (k=1,2) \tag{31}$$

where

$$\psi^1(\mu') = \tfrac{3}{8}(1-\mu'^2)(1+2\mu'^2) \tag{32}$$

$$\psi^2(\mu') = \tfrac{3}{16}(1+\mu'^2) \tag{33}$$

and

$$\mathbf{S}^{(k)}(\tau_1;\Omega,\Omega_0) = \mathbf{P}^{(k)}(\mu,-\mu_0;k(\phi_0-\phi))S_k(\tau_1;\mu,\mu_0). \tag{34}$$

The reciprocity relations for S_k and T_k are

$$S_k(\tau_1;\mu,\mu') = S_k(\tau_1;\mu',\mu)$$

and (35)

$$T_k(\tau_1;\mu,\mu') = T_k^*(\tau_1;\mu',\mu)$$

where T_k^* is obtained from equation (31) by interchanging μ and μ_0. The integro-differential equation for the azimuth independent term can be simplified slightly by treating the Stokes parameter V separately and the matrix $\mathbf{P}^{(0)}(\mu,\mu_0)$ in equation (25) can be written

$$\mathbf{P}^{(0)}(\mu,\mu') = \tfrac{3}{4}\begin{pmatrix} 2(1-\mu^2)(1-\mu'^2)+\mu^2\mu'^2 & \mu^2 \\ \mu'^2 & 1 \end{pmatrix} \tag{36}$$

The matrix $\mathbf{P}^{(0)}$ can be factored in the following way

$$\mathbf{P}^{(0)}(\mu, \mu') = \tfrac{3}{4}\mathbf{M}(\mu)\tilde{\mathbf{M}}(\mu') \tag{37}$$

where

$$\mathbf{M}(\mu) = \begin{pmatrix} \mu^2 & \sqrt{2}(1-\mu^2) \\ 1 & 0 \end{pmatrix} \tag{38}$$

and the tilde denotes matrix transposition. With this notation the integro-differential equation for $\mathbf{S}^{(0)}$ becomes

$$\begin{aligned} \left\{\frac{1}{\mu_0}+\frac{1}{\mu}+\frac{\partial}{\partial\tau_1}\right\} \mathbf{S}^{(0)}(\tau_1;\mu,\mu_0) &= \frac{3}{4}\lambda(\tau_1)\left\{\mathbf{M}(\mu)+\frac{1}{2}\int_0^1 \mathbf{S}^{(0)}(\tau_1;\mu,\mu')\mathbf{M}(\mu')\frac{d\mu'}{\mu'}\right\} \\ &\quad\times\left\{\tilde{\mathbf{M}}(\mu_0)+\frac{1}{2}\int_0^1 \tilde{\mathbf{M}}(\mu'')\mathbf{S}^{(0)}(\tau_1;\mu'',\mu_0)\frac{d\mu''}{\mu''}\right\} \end{aligned} \tag{39}$$

$$\begin{aligned} \left\{\frac{1}{\mu_0}+\frac{\partial}{\partial\tau_1}\right\} \mathbf{T}^{(0)}(\tau_1;\mu,\mu_0) &= \frac{3}{4}\lambda(\tau_1)\left\{\mathbf{M}(\mu)\,e^{-\tau_1/\mu}+\frac{1}{2}\int_0^1 \mathbf{T}^{(0)}(\tau_1;\mu,\mu')\mathbf{M}(\mu')\frac{d\mu'}{\mu'}\right\} \\ &\quad\times\left\{\tilde{\mathbf{M}}(\mu)+\frac{1}{2}\int_0^1 \tilde{\mathbf{M}}(\mu')\mathbf{S}^{(0)}(\tau_1;\mu,\mu')\frac{d\mu'}{\mu'}\right\}. \end{aligned} \tag{40}$$

The reciprocity relations can be introduced to reduce the number of equations as follows:

$$\begin{aligned} \mathbf{S}^{(0)}(\tau_1;\mu,\mu') &= \tilde{\mathbf{S}}^{(0)}(\tau_1;\mu',\mu) \\ \mathbf{T}^{(0)}(\tau_1;\mu,\mu') &= \tilde{\mathbf{T}}^{(0)*}(\tau_1;\mu',\mu), \end{aligned} \tag{41}$$

where $\tilde{\mathbf{T}}^{(0)*}$ satisfies an equation obtained by transposing (40) and interchanging μ and μ_0.

By replacing the integrals by Gauss sums of order M the number of equations to be solved now taking reciprocity into account is $2M(3M+2)$ as compared to $3M(M+1)$ for the scalar case for Rayleigh scattering.

Once $\mathbf{S}$ and $\mathbf{T}$ have been determined then the reflected and transmitted intensity is easily obtained by

$$\mathbf{I}(0;+\mu,\phi) = \frac{1}{4\mu}\mathbf{S}(\tau_1;\mu,\phi;\mu_0,\phi_0)\mathbf{F} \tag{42}$$

$$\mathbf{I}(\tau_1;-\mu,\phi) = \frac{1}{4\mu}\mathbf{T}(\tau_1;\mu,\phi;\mu_0,\phi_0)\mathbf{F} \tag{43}$$

The addition of a Lambert surface at the lower boundary of the atmosphere and the resulting change in reflected and transmitted intensities may easily be calculated from the previously computed functions (CHANDRASEKHAR[1]).

COMPUTATIONAL METHODS AND RESULTS

The resulting coupled non-linear integro-differential equations were solved using a combination of numerical techniques to obtain the greatest possible accuracy in calculating the Stokes' vector. The final code written, using double precision arithmetic on the IBM 360/50, is capable of computing results for large optical depths in a relatively short time for any ground albedo less than unity and any single scattering albedo which may vary with optical depth. MULLIKIN[2] has shown that singular solutions exist near the desired solutions and that one must carry high precision in the numerical integration.

The integrals with respect to μ are approximated by Gauss quadrature normalized to the interval 0–1. Thus, the functions continuous in μ and μ_0 are replaced by a set of functions evaluated for values of μ and μ_0 determined by the order of the Gauss quadrature, M. The integration with respect to τ is performed by using Runge–Kutta integration as a starting procedure and then switching to the fifth order Adams–Bashforth predictor coupled with the fifth order Adams–Moulton corrector. To test the numerical procedure, the integro-differential equations of Chandrasekhar's X- and Y-functions for an isotropic phase function were solved. These results are shown in Table 1 with the numerical results of

TABLE 1. COMPARISON OF THE INVARIANT IMBEDDING APPROACH AND THE INTEGRAL EQUATION APPROACH FOR THE COMPUTATION OF THE X AND Y FUNCTIONS FOR ISOTROPIC SCATTERING WITH $\mu_0 = 0.50$ AND $\omega_0 = 1.0$

τ		Present scheme	CARLSTEDT–MULLIKIN[7]	BELLMAN *et al.*[4]
0.2	X	1.24475	1.24480	1.24456
	Y	8.98554-1	8.98582-1	8.98214-1
0.4	X	1.37253	1.37252	1.37252
	Y	7.66784-1	7.66758-1	7.66827-1
0.6	X	1.45999	1.46000	1.46003
	Y	6.57025-1	6.57032-1	6.57095-1
0.8	X	1.52442	1.52442	1.52443
	Y	5.69441-1	5.69437-1	5.69436-1
1.0	X	1.57403	1.57404	1.57403
	Y	5.00054-1	5.00045-1	5.00032-1
1.2	X	1.61352	1.61352	1.61352
	Y	4.44882-1	4.44875-1	4.44872-1
1.4	X	1.64578	1.64578	1.64578
	Y	4.00627-1	4.00621-1	4.00626-1
1.6	X	1.67272	1.67272	1.67272
	Y	3.64711-1	3.64707-1	3.64715-1
1.8	X	1.69561	1.69561	1.69561
	Y	3.35177-1	3.35174-1	3.35181-1
2.0	X	1.71535	1.71535	1.71535
	Y	3.10555-1	3.10552-1	3.10557-1
2.2	X	1.73260	1.73260	1.73260
	Y	2.89744-1	2.89742-1	2.89745-1
2.4	X	1.74783	1.74783	1.74783
	Y	2.71921-1	2.71919-1	2.71921-1
2.6	X	1.76140	1.76140	1.76140
	Y	2.56467-1	2.56465-1	2.56467-1
2.8	X	1.77358	1.77358	1.77358
	Y	2.42915-1	2.42913-1	2.42914-1
3.0	X	1.78459	1.78459	1.78459
	Y	2.30908-1	2.30907-1	2.30908-1
3.5	X	1.80803	1.80803	1.80803
	Y	2.06009-1	2.06008-1	2.06009-1

CARLSTEDT and MULLIKIN[7] and BELLMAN *et al.*[3] The results of BELLMAN *et al.*[3] were calculated by a similar technique using invariant imbedding. As can be seen from this table the agreement is excellent. We also found that a change in order from 7 to 9 in the Gaussian integration results in a change of at most 1 part in 3×10^5 in total intensities for $\mu = \mu_0 = 0.5000$ for all optical depths considered. All results computed and shown in this paper were done with a Gauss quadrature of order 9 so that μ values near those computed by COULSON *et al.*[9] could be obtained for comparison.

A check of results from the calculations of the Stokes' vector in the matrix case was also accomplished. CHANDRASEKHAR[1] has shown that $\mathbf{S}^{(0)}(\tau; \mu, \mu_0)$ and $\mathbf{T}^{(0)}(\tau; \mu, \mu_0)$ must be expressible in terms of functions of μ and μ_0. These functions are $\psi(\mu)$, $\phi(\mu)$, $\chi(\mu)$, $\zeta(\mu)$, $\xi(\mu)$, $\eta(\mu)$, $\sigma(\mu)$, $\theta(\mu)$, $\gamma_l^{(1)}(\mu)$, and $\gamma_r^{(1)}(\mu)$, where:

$$\psi(\mu) = \mu^2 + \frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}[\mu'^2 S_{ll}^{(0)}(\mu, \mu') + S_{lr}^{(0)}(\mu, \mu')],$$

$$\phi(\mu) = 1 - \mu^2 + \frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}(1 - \mu'^2) S_{ll}^{(0)}(\mu, \mu'),$$

$$\chi(\mu) = 1 + \frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}[\mu'^2 S_{rl}^{(0)}(\mu, \mu') + S_{rr}^{(0)}(\mu, \mu')],$$

$$\zeta(\mu) = \frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}(1 - \mu'^2) S_{rl}^{(0)}(\mu, \mu'),$$

$$\xi(\mu) = \mu^2 e^{-\tau_1/\mu} + \frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}[\mu'^2 T_{ll}^{(0)}(\mu, \mu') + T_{lr}^{(0)}(\mu, \mu')],$$

$$\eta(\mu) = (1 - \mu^2) e^{-\tau_1/\mu} + \frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}(1 - \mu'^2) T_{ll}^{(0)}(\mu, \mu'),$$

$$\sigma(\mu) = e^{-\tau_1/\mu} + \frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}[\mu'^2 T_{rl}^{(0)}(\mu, \mu') + T_{rr}^{(0)}(\mu, \mu')],$$

$$\theta(\mu) = \frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}(1 - \mu'^2) T_{rl}^{(0)}(\mu, \mu'),$$

$$t_l(\mu) = \frac{1}{2}\int_0^1 [T_{ll}^{(0)}(\mu', \mu) + T_{rl}^{(0)}(\mu', \mu)]\,d\mu',$$

$$t_r(\mu) = \frac{1}{2}\int_0^1 [T_{lr}^{(0)}(\mu', \mu) + T_{rr}^{(0)}(\mu', \mu)]\,d\mu',$$

$$\gamma_l^{(1)}(\mu) = \frac{t_l(\mu)}{\mu} + e^{-\tau_1/\mu},$$

$$\gamma_r^{(1)}(\mu) = \frac{t_r(\mu)}{\mu} + e^{-\tau_1/\mu},$$

and

$$\bar{S} = \int_0^1\int_0^1 S^{(0)}(\mu', \mu'')\,d\mu'\,d\mu''.$$

These functions are in a form slightly different from those of CHANDRASEKHAR[1] due to the difference in the explicit expression for $\mathbf{I}^{(0)}(0, \mu, \mu_0)$ and $\mathbf{I}^{(0)}(\tau_1, -\mu, \mu_0)$. Shown also in Table 2, are the resulting calculations obtained by CHANDRASEKHAR and ELBERT.[8] The agreement between corresponding functions for an optical depth of $\tau = 0.02$ is very good while noticeable differences occur for optical depths of 0.5 and 1.0. In the discussion of their results, Chandrasekhar and Elbert note for increasing optical depths their values lacked stability and were subject to more error than for small optical depths.

A final check of the numerical results was obtained by calculating the diffusely reflected and transmitted flux normal to both the upper and lower boundaries in the scalar and matrix cases. The sum of the upward flux F_u, the downward flux F_d, and the direct solar flux $F_s = \mu_0\pi\,e^{-\tau/\mu_0}F$ is equal to $\mu_0\pi F$ in the absence of absorption in the atmosphere. With this notation $(F_u + F_d + F_s)/\mu_0\pi F = 1.0$. The resulting calculations are shown in Table 3 and agreement is quite good. The close agreement of the upward and downward fluxes between the scalar and matrix approach should be noted. The fluxes were obtained by integrating the azimuth independent part of the intensity over μ. In both cases for the two optical depths shown, the relative stability of the two codes for increasing optical depth may be noted. The largest error occurs for small values of μ_0 which is to be expected. The propagation of round off error does not seem to be significant in either case.

DISCUSSION

In Figs. 1 and 2, the reflected intensities for both the scalar and vector solutions are plotted as a function of μ for $\phi = 0$ and $\phi = \pi/2$ respectively with $A = 0$ for some selected values of both μ_0 and τ. Since it is difficult to read accurate values from graphs, we developed Table 4 which gives the percentage error between the two calculations. This error is defined as $(I_v - I_s)100/I_v$ where I_v is the intensity computed from the correct vector field approach and I_s is the intensity computed from the scalar theory. As can be seen from this table the percentage error can be quite significant reaching a maximum value of approximately 12 per cent in absolute value for $\mu, \mu_0 = 0.984$ with $\phi = 0$ and $\tau = 1$. For fixed values of μ and μ_0

TABLE 2. THE INTEGRAL FUNCTIONS OF μ COMPUTED BY PRESENT SCHEME COMPARED WITH VALUES COMPUTED BY CHANDRASEKHAR FOR $\mu = 0.500$ AND $\omega_0 = 1.00$

$\tau = 0.02$	Present scheme	CHANDRASEKHAR	$\tau = 0.50$	Present scheme	CHANDRASEKHAR	$\tau = 1.00$	Present scheme	CHANDRASEKHAR
$\bar{s}$	0.0190970	0.01904	$\bar{s}$	0.2960391	0.2931	$\bar{s}$	0.4469077	0.4401
ψ	0.2618474	0.26176	ψ	0.4103808	0.41132	ψ	0.4980401	0.49868
ϕ	0.7935638	0.79302	ϕ	1.0830593	1.08010	ϕ	1.1662154	1.15646
χ	1.0355556	1.03517	χ	1.3227471	1.32297	χ	1.4339508	1.43443
ζ	0.0019679	0.00197	ζ	0.0502914	0.05030	ζ	0.0855749	0.08512
ξ	0.2520076	0.25192	ξ	0.2311753	0.22980	ξ	0.2148163	0.20563
η	0.7636574	0.76340	η	0.5297891	0.53607	η	0.3238092	0.35580
σ	0.9960150	0.99582	σ	0.6312118	0.63397	σ	0.4114063	0.42160
θ	0.0019639	0.00197	θ	0.0461865	0.04499	θ	0.0679823	0.06282
$\gamma_l^{(1)}$	0.9803653	0.98039	$\gamma_l^{(1)}$	0.6637841	0.66825	$\gamma_l^{(1)}$	0.4969555	0.50734
$\gamma_r^{(1)}$	0.9803766	0.98039	$\gamma_r^{(1)}$	0.6669395	0.66803	$\gamma_r^{(1)}$	0.5046866	0.50708

TABLE 3. COMPARISON OF FLUXES FOR BOTH THE SCALAR AND MATRIX CASE FOR RAYLEIGH SCATTERING

τ	μ_0	Case	F_u	F_d	$(F_u+F_d+F_s)/\mu_0\pi F$
1.00	0.01592	Scalar	0.03762	0.01241	1.00058
		Matrix	0.03769	0.01234	1.00041
	0.50000	Scalar	0.78381	0.57447	1.00004
		Matrix	0.78416	0.57410	1.00004
	0.98408	Scalar	1.06376	0.90882	1.00003
		Matrix	1.06286	0.90968	1.00001
5.00	0.01592	Scalar	0.04543	0.00459	1.00021
		Matrix	0.04548	0.00454	1.00011
	0.50000	Scalar	1.28741	0.28342	1.00006
		Matrix	1.28827	0.28251	1.00004
	0.9841	Scalar	2.28850	0.78412	1.00008
		Matrix	2.28588	0.78660	1.00004

TABLE 4. PER CENT ERRORS BETWEEN THE VECTOR AND SCALAR REFLECTED INTENSITIES FOR VARIOUS μ_0, μ, AND τ FOR $\phi = 0$ AND $\pi/2$ WITH $A = 0$

μ	$\mu_0 = 0.1933, \phi = 0$				$\mu_0 = 0.1933, \phi = \pi/2$			
	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$
0.0159	4.32	4.66	4.72	4.63	0.074	0.110	0.064	−0.004
0.1933	5.74	6.46	6.50	6.24	0.207	0.408	0.374	0.256
0.5000	2.77	3.80	4.00	3.67	−1.81	−1.68	−1.59	−1.52
0.8067	−5.13	−6.23	−4.76	−3.98	−6.03	−5.87	−5.57	−4.76
0.9841	−11.3	−11.7	−10.1	−8.11	−9.77	−9.30	−8.92	−7.18

μ	$\mu_0 = 0.500, \phi = 0$				$\mu_0 = 0.500, \phi = \pi/2$			
	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$
0.0159	3.87	4.62	4.77	4.51	−1.79	−1.90	−1.91	−1.87
0.1933	2.77	3.80	4.00	3.67	−1.81	−1.71	−1.59	−1.52
0.5000	−2.63	−2.11	−1.44	−1.13	−2.37	−2.41	−2.14	−1.85
0.8067	−10.1	−10.5	−8.84	−6.71	−3.33	−3.81	−3.44	−2.72
0.9841	−7.02	−8.63	−7.78	−5.81	−4.06	−4.90	−4.48	−3.40

μ	$\mu_0 = 0.8067, \phi = 0$				$\mu_0 = 0.8067, \phi = \pi/2$			
	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$
0.0159	−1.20	−0.856	−0.546	−0.495	−5.491	−5.74	−5.36	−4.76
0.1933	−5.13	−4.76	−3.98	−3.34	−6.03	−6.24	−5.57	−4.74
0.5000	−10.1	−10.5	−8.84	−6.71	−3.33	−3.81	−3.44	−2.72
0.8067	−7.48	−9.02	−8.28	−5.98	1.05	0.505	0.035	−0.027
0.9841	2.23	0.653	−0.497	−0.540	3.82	3.40	2.42	1.67

μ	$\mu_0 = 0.9841, \phi = 0$				$\mu_0 = 0.9841, \phi = \pi/2$			
	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$
0.0159	−8.27	−8.52	−7.66	−6.47	−8.75	−9.04	−8.15	−6.87
0.1933	−11.3	−11.7	−10.14	−8.11	−9.77	−10.3	−8.92	−7.18
0.5000	−7.02	−8.6	−7.78	−5.81	−4.06	−4.90	−4.48	−3.40
0.8067	2.23	0.65	−0.497	−0.540	3.82	3.40	2.42	1.67
0.9841	7.50	7.44	5.86	4.03	8.16	8.30	6.76	4.74

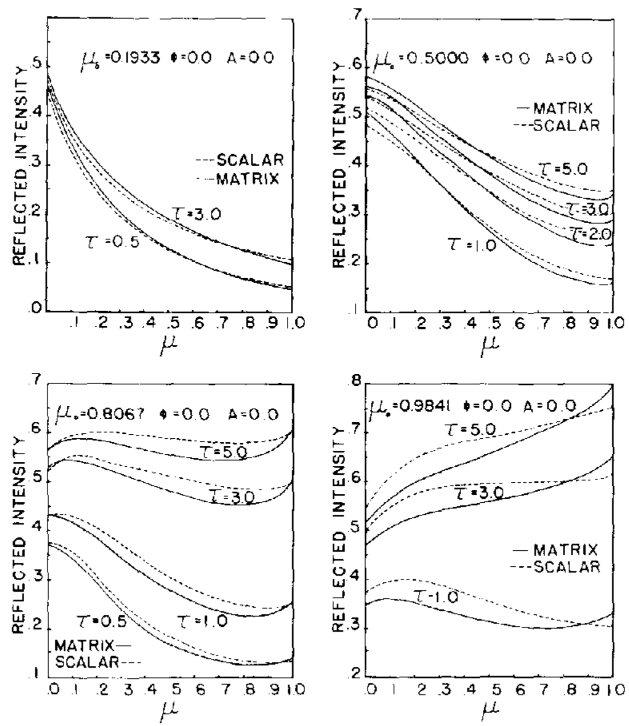

FIG. 1. Reflected intensities as a function of μ for both the vector and scalar approach for several values of μ_0 and τ with $\phi = 0$ and $A = 0$.

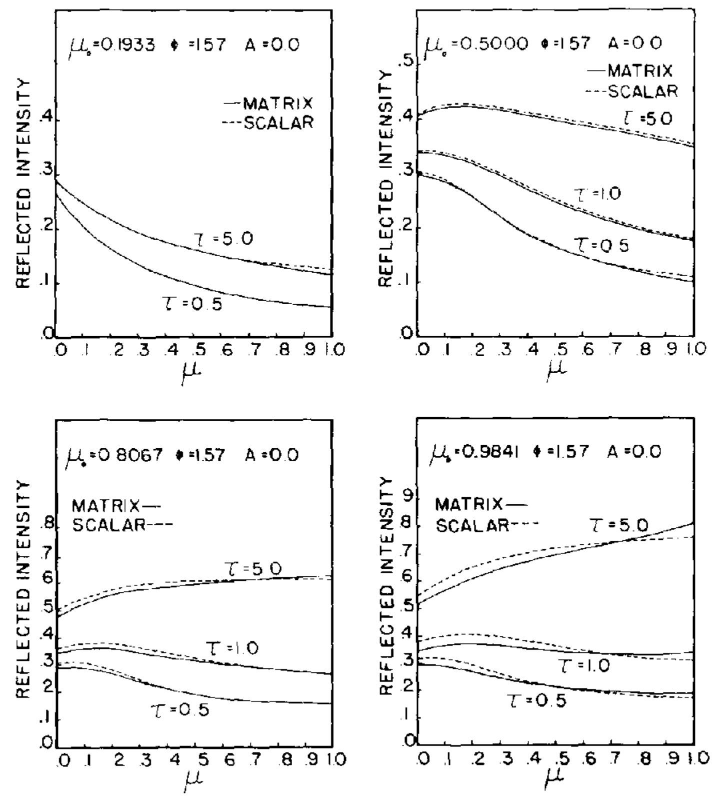

FIG. 2. Reflected intensities as a function of μ for both the vector and scalar approach for several values of μ_0 and τ with $\phi = \pi/2$ and $A = 0$.

the error as a function of τ tends to reach a maximum and decrease monotonically to some asymptotic value for $\tau \to \infty$. This behavior is entirely expected since as the atmosphere becomes very optically thick the very high order collisions give smaller contributions to the total. For corresponding values of μ, μ_0 and τ the percentage error is almost always smaller in absolute value for $\phi = \pi/2$ than for $\phi = 0$.

In Figs. 3 and 4 the corresponding transmitted intensities are plotted. Table 5 gives the corresponding percentage errors in the transmitted intensity. It should be noted that errors in excess of 17 per cent can be encountered by using the incorrect scalar theory. Of course as τ becomes large the percentage error will decrease since the photons reaching the bottom will have undergone thousands of collisions and polarization effects are virtually destroyed.

The cases for $A \neq 0$ were not presented since in general the percentage errors are smaller when a Lambert surface is inserted at the bottom of the atmosphere. This is due to the fact that a Lambert surface sends up unpolarized radiation in all directions with equal intensity.

Table 5. Per cent errors between the vector and scalar transmitted intensities for various μ_0, μ, and τ for $\phi = 0$ and $\pi/2$ with $A = 0$

μ	$\mu_0 = 0.1933, \phi = 0$				$\mu_0 = 0.1933, \phi = \pi/2$			
	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$
0.0159	13.3	17.3	11.2	−0.124	−1.87	−4.61	−6.53	−3.74
0.1933	10.5	16.2	12.0	−0.399	−0.244	−1.62	−4.41	−3.15
0.5000	7.86	11.2	10.6	−0.937	−2.18	−2.63	−3.27	−2.24
0.8067	2.14	3.44	3.95	−0.586	−6.55	−7.03	−5.17	−1.58
0.9841	−7.73	−8.17	5.32	1.01	−10.4	−11.1	−7.45	−1.45

μ	$\mu_0 = 0.5000, \phi = 0$				$\mu_0 = 0.5000, \phi = \pi/2$			
	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$
0.0159	6.60	9.60	8.78	0.424	−3.00	−4.10	−4.82	−2.79
0.1933	7.86	11.2	10.64	0.928	−2.18	−2.63	3.27	2.24
0.5000	8.45	11.3	11.4	1.85	−2.55	−2.85	−2.68	−1.42
0.8067	6.26	7.90	7.71	1.89	−3.47	−4.14	3.43	0.853
0.9841	−0.581	−0.964	−0.703	0.106	−4.21	−5.23	−4.25	−0.682

μ	$\mu_0 = 0.8067, \phi = 0$				$\mu_0 = 0.8067, \phi = \pi/2$			
	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$
0.0159	−0.720	0.288	1.32	−0.149	−6.94	−7.43	−5.70	−2.00
0.1933	2.14	3.44	3.95	0.583	−6.55	−7.03	−5.17	−1.58
0.5000	6.26	7.90	7.71	1.89	−3.47	−4.14	−3.43	−0.853
0.8067	8.23	9.66	8.80	2.60	1.04	0.399	−0.309	−0.073
0.9841	6.21	6.45	4.99	1.46	3.87	3.40	1.99	0.555

μ	$\mu_0 = 0.9841, \phi = 0$				$\mu_0 = 0.9841, \phi = \pi/2$			
	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$	$\tau = 0.5$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 5.0$
0.0159	−9.55	−9.65	−6.40	−1.50	−10.5	−10.7	−7.22	1.71
0.1933	−7.73	−8.17	−5.33	−1.02	−10.4	−11.1	−7.45	−1.45
0.5000	−0.581	−0.964	−0.703	0.106	−4.21	−5.23	−4.26	−0.682
0.8067	6.21	6.45	4.99	1.461	3.86	3.40	1.99	0.555
0.9841	8.90	9.33	7.19	2.05	8.26	8.45	6.25	1.71

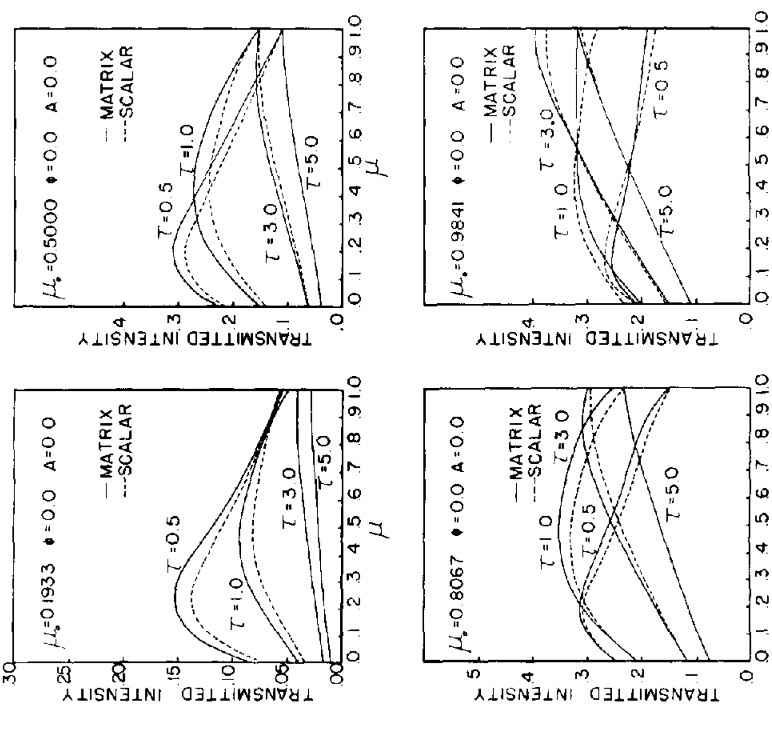

FIG. 3. Transmitted intensities as a function of μ for both the vector and scalar approach for several values of μ_0 and τ with $\phi = 0$ and $A = 0$.

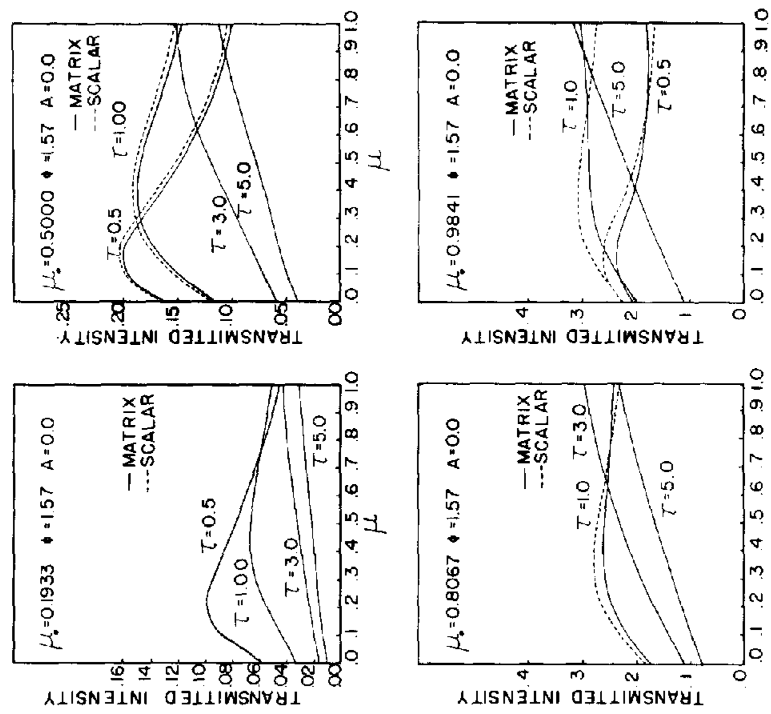

FIG. 4. Transmitted intensities as a function of μ for both the vector and scalar approach for several values of μ_0 and τ with $\phi = \pi/2$ and $A = 0$.

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